3-regular subgraphs of 4-regular graphs

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Let G be a 4-regular (multi)graph. Does G contain an edge-induced 3-regular subgraph? Not necessarily, as the example of a triangle with two edges between each vertex shows. Now suppose we add an arbitrary edge to G. It turns out that the claim is now true. This was proved by Alon, Friedland, and Kalai [AFK84a] using the following clever argument.

First we need a fact from number theory:

Theorem 1 *(Chevalley-Waring) Let* $\{P_i(x_1, \ldots, x_m)\}$ *be a system of polynomials over* $\mathbf{F}_p$ *such that the sum* D *of the total degrees of the polynomials is less than* m*. Then the number of zeros of the system is congruent to* 0 *modulo* p*; in particular, if the system has one zero, then it has at least two.*

Proof Let

$$Q(x_1, \ldots, x_m) = \prod_i \left(1 - P_i(x_1, \ldots, x_m)^{p-1}\right).$$

By Fermat's Little Theorem, Q is 1 exactly where all P_i's are zero, and 0 everywhere else. Then it is enough to prove that

$$\sum_{(x_1, \ldots, x_m) \in \mathbf{F}_p^m} Q(x_1, \ldots, x_m) = 0.$$

The trick is to fix a primitive root ζ of $\mathbf{F}_p$ and view the summation as indexed according to ζ^i. Let $cx_1^{z_1} \cdots x_m^{z_m}$ be any term of Q. Then

$$\sum_{(x_1, \ldots, x_m) \in \mathbf{F}_p^m} cx_1^{z_1} \cdots x_m^{z_m} = \sum_{(y_1, \ldots, y_m) \in \{0, \ldots, p-2\}^m} c\zeta^{y_1 z_1} \cdots \zeta^{y_m z_m} = c\left(\sum_{y_1} \zeta^{y_1 z_1}\right) \cdots \left(\sum_{y_m} \zeta^{y_m z_m}\right).$$

The total degree of Q is $(p-1)D < (p-1)m$, so some z_i above must be less than $p-1$. If $z_i = 0$ then $\sum_{y_i} \zeta^{y_i z_i} = p = 0$; if $0 < z_i < p-1$ then $\zeta^{z_i} \neq 1$ and $\sum_{y_i} \zeta^{y_i z_i} = \sum_{y_i} \zeta^{(y_i+1) z_i} = \zeta^{z_i} \sum_{y_i} \zeta^{y_i z_i}$, which is possible only if $\sum_{y_i} \zeta^{y_i z_i} = 0$. In either case, we have that each term of Q sums to zero, so all of Q sums to zero. ■

Now associate with each edge e of G (including the added edge) a variable $x_e \in \mathbf{F}_3$. Consider the system of equations formed by associating with each vertex v of G the equation

$$\sum_{e \text{ incident to } v} x_e^2 \equiv 0 \pmod 3.$$

Let $\{x_e\}$ be any solution to this system, and set $H = \{e : x_e \neq 0\}$. If H is non-empty, then the subgraph induced by H is 3-regular, since the number of edges in H incident to any vertex is divisible by 3, greater than 0, and at most 5, which allows only 3.

It remains to show that there exists a non-trivial solution to this system. Each equation has degree 2, so the sum of degrees is $2n$, where n is the number of vertices. The number of variables equals the number of edges in a 4-regular graph plus one edge, which is $2n + 1$. So the sum of total degrees is strictly less than the number of variables. Further, there is one solution to the system (the all-zeros solution). By Theorem 1, there must exist at least one other solution, which is not the trivial one.

Using a more sophisticated version of this argument, Alon, Friedland, and Kalai [AFK84b] proved the following.

Theorem 2 *Suppose that every vertex of a graph G has degree k or $k+1$, and at least one vertex has degree $k+1$. Then for every prime power q such that $k \geq 2(q-1)$, G contains a q-regular subgraph.*

References

[AFK84a] N. Alon, S. Friedland, and G. Kalai. Every 4-regular graph plus an edge contains a 3-regular subgraph. *Journal of Combinatorial Theory Series B*, 37:92–93, 1984.

[AFK84b] N. Alon, S. Friedland, and G. Kalai. Regular subgraphs of almost regular graphs. *Journal of Combinatorial Theory Series B*, 37:79–91, 1984.